Exploring Conjugation with GAP

Two elements *g* and *g'* in a group *G* are said to be conjugate if $g = h * g' * h^{-1}$ (or $g' = h * g * h^{-1}$) for some *h* in *G* . For example, if you execute the code below you will find that the elements *g1* = (1, 2, 3, 4)(7, 8, 9) and *g2* = (1, 4, 3, 2)(5, 7, 8) are conjugate in S_{10}.

```
gap> G := SymmetricGroup(10);
gap> g1 := (1, 2, 3, 4)(7, 8, 9);  h := (2,4)(5,9);
gap> h * g1 * h^(-1);
```

In this lab we will use **gap** to explore the relationship between an element and its conjugate. In particular, we will explore such questions as: Given any two elements *g1* and *g2* in a group, can you always find an element *h* such that $g2 = h * g1 * h^{-1}$? If not, then when are two elements in a permutation group conjugate? Our approach, initially, will be experimental; we will make conjectures based on **gap**'s output. For homework you will then be asked to provide rigorous proofs of your findings.

Question 1. Let $G = S_{10}$ and assume h1, h2, h3, h4, and h5 are defined to be:

```
gap> h1: = (1, 2, 3);
gap> h2: = (2, 3)(5, 4, 7);
gap> h3: = (1, 2, 3)(8, 9, 10);
gap> h4: = (1, 2, 4)(5, 9);
gap> h5: = (4, 6, 7, 9);
```

Complete the tables that follow.

g1	(1, 2, 3, 4)(7, 8, 9)
$h1*g1*h1^{-1}$	
$h2*g1*h2^{-1}$	
$h3*g1*h3^{-1}$	
$h4*g1*h4^{-1}$	
$h5*g1*h5^{-1}$	

g2	(1, 4, 3, 2)(5, 7, 8)
$h1*g2*h1^{-1}$	
$h2*g2*h2^{-1}$	
$h3*g2*h3^{-1}$	
$h4*g2*h4^{-1}$	
$h5*g2*h5^{-1}$	

g3	(1, 3)(7, 8, 9)
$h1*g3*h1^{-1}$	
$h2*g3*h2^{-1}$	
$h3*g3*h3^{-1}$	
$h4*g3*h4^{-1}$	
$h5*g3*h5^{-1}$	

g4	(1, 7)
$h1*g4*h1^{-1}$	
$h2*g4*h2^{-1}$	
$h3*g4*h3^{-1}$	
$h4*g4*h4^{-1}$	
$h5*g4*h5^{-1}$	

What does the output suggest? Would you expect the permutations *g5* = (1, 2, 5, 6) and *g6* = (2, 3)(4, 5) to be conjugate? Why or why not?

Question 2. GAP will actually allow you to look at all of the elements in S_{10} that are conjugate to any given element. For example, to get the list of conjugates of *g5* you execute the following. (Be patient! This computation will likely take a while.)

```
gap> g5 := (1, 2, 5, 6);
gap> c:= ConjugacyClass(G, g5);
gap> Elements(c);
```

Note: The command "ConjugacyClass" refers to the fact that conjugation is an equivalence relation on the set of group elements. More specifically, we say that $[g] = \{h*g*h^{-1} \mid h \in G\}$ is the equivalence class or *conjugacy class* of *g*.

What does the output of this code seem to indicate? What can you say about the cycle structure of *g5* compared to the cycle structure of each of its conjugates?

Question 3. Use **gap**'s "Size" command to determine whether or not every four cycle in S_{10} is conjugate to *g5*. Explain your reasoning.

Question 4. To summarize your findings, provide a *conjecture* about the relationship between a permutation $g \in S_{10}$ and any conjugate $h*g*h^{-1}$, where $h \in S_{10}$. *Your homework assignment for next lesson is to prove (or disprove) this conjecture!*

Conjecture:

Question 5. How many distinct conjugacy classes does S_4 have?

Question 6. How many distinct conjugacy classes does S_5 have?

Returning to the Rubik's Cube…
Recall that the generators of the group of configurations of the Rubik's Cube were described as:

```
F:=(17,19,24,22)(18,21,23,20)(6,25,43,16)(7,28,42,13)(8,30,41,11);

R:=(25,27,32,30)(26,29,31,28)(3,38,43,19)(5,36,45,21)(8,33,48,24);

U:=(1,3,8,6)(2,5,7,4)(9,33,25,17)(10,34,26,18)(11,35,27,19);

B:=(33,35,40,38)(34,37,39,36)(3,9,46,32)(2,12,47,29)(1,14,48,27);

D:=(41,43,48,46)(42,45,47,44)(22,30,38,14)(23,31,39,15)(24,32,40,16);

L:=(9,11,16,14)(10,13,15,12)(1,17,41,40)(4,20,44,37)(6,22,46,35);
```

Are you thinking that these generators must all belong to the same conjugacy class? You should be!

Question 7. Find the permutation *h* in the cube group having the property that $U = h*F*h^{-1}$. That is, prove that *U* and *F* are indeed conjugate. Hint: You'll want to think about this problem from a geometric point of view. Then use **gap** to prove algebraically that your candidate for *h* works. The diagram below might be helpful.

			1	2	3						
			4	**Up**	5						
			6	7	8						
9	10	11	17	18	19	25	26	27	33	34	35
12	**Left**	13	20	**Front**	21	28	**Right**	29	36	**Back**	37
14	15	16	22	23	24	30	31	32	38	39	40
			41	42	43						
			44	**Down**	45						
			46	47	48						

You will need (want?) to read in the file "rubik.gap" before defining the cube group. Recall that this file served to define the generators given above.

gap> Read("rubik.gap");
gap> G:=SymmetricGroup(48);
gap> cube:=Subgroup(G, [F, B, L, R, U, D]);

Question 8. Provide a geometric description of all elements in the conjugacy class of $(R^2D^2)^3$ = (21, 36)(23, 39)(28, 29)(42, 47).