Quantum Information Theory II: States and Dynamics

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Reference: B. Schumacher, *Phys. Rev. A* **54**, pp. 2614–2628 (1996). See sections II, III and the Appendix.

Pure states

A pure state of Q is a vector $\left|\phi^Q\right\rangle$in $\mathcal{H}^Q$.

- $\left\langle \phi^Q \middle| \psi^Q \right\rangle = 1.$
- Given an orthonormal basis $\left|a_k^Q\right\rangle$,

$$\left|\phi^Q\right\rangle = \sum_k a_k \left|a_k^Q\right\rangle$$

where $a_k = \left\langle a_k^Q \middle| \phi^Q \right\rangle$

Pure states of a compound system RQ

- $\left|\Psi^{RQ}\right\rangle \in \mathcal{H}^{RQ} = \mathcal{H}^R \otimes \mathcal{H}^Q$
- Product state $\left|\Psi^{RQ}\right\rangle = \left|\alpha^R\right\rangle \otimes \left|\beta^Q\right\rangle$
(We often write this as $\left|\alpha^R \beta^Q\right\rangle$.)
- Entangled state $\left|\Psi^{RQ}\right\rangle$

$$\left|\Psi^{RQ}\right\rangle = \sum_{km} c_{km} \left|a_k^R b_m^Q\right\rangle$$

Schmidt decomposition

(a/k/a "polar" decomposition)

Given an entangled state $\left|\Psi^{RQ}\right\rangle$, we can always find orthonormal bases $\left|a_k^R\right\rangle$ for R and $\left|b_k^Q\right\rangle$ for Q so that

$$\boxed{\left|\Psi^{RQ}\right\rangle = \sum_k c_k \left|a_k^R b_k^Q\right\rangle}$$

where c_k is real and non-negative.

The superposition is "real and diagonal" with respect to this product basis.

Observables

Complete measurements

- Basis of "eigenstates" $\left| a_k^Q \right\rangle$
- Outcomes $p_k = \left\langle \psi^Q \left| a_k^Q \right\rangle \left\langle a_k^Q \right| \psi^Q \right\rangle$
- "Eigenvalues" a_k yield operator A^Q

$$\begin{aligned} A^Q &= \sum_k a_k \left| a_k^Q \right\rangle \left\langle a_k^Q \right| \\ \langle A \rangle &= \left\langle \psi^Q \right| A^Q \left| \psi^Q \right\rangle \end{aligned}$$

Incomplete measurements

- Projections Π_k^Q with $\sum_k \Pi_k^Q = 1^Q$
- Outcomes $p_k = \left\langle \psi^Q \right| \Pi_k^Q \left| \psi^Q \right\rangle$
- $A^Q = \sum_k a_k \Pi_k^Q$, etc.

Dynamics

Schrödinger equation

$$H^Q \left|\psi^Q\right\rangle = i\hbar \frac{d}{dt} \left|\psi^Q\right\rangle$$

Time-evolution operator U^Q

$$\left|\psi^Q(t)\right\rangle = U^Q(t) \left|\psi^Q(0)\right\rangle$$

where $U^Q = \exp\left(\frac{1}{i\hbar} H^Q t\right)$ is unitary:

$$U^{Q\dagger} U^Q = 1^Q$$

Conservation of inner product

$$\begin{aligned} \left\langle \psi_1^{Q'} \middle| \psi_2^{Q'} \right\rangle &= \left\langle \psi_1^Q \middle| U^{Q\dagger} U^Q \middle| \psi_2^Q \right\rangle \\ &= \left\langle \psi_1^Q \middle| \psi_2^Q \right\rangle \end{aligned}$$

Trace of an operator

Choose some orthonormal basis $\left|a_k^Q\right\rangle$.

The **trace** of an operator B^Q is

$$\mathrm{Tr}\, B^Q = \sum_k \left\langle a_k^Q \right| B^Q \left| a_k^Q \right\rangle$$

$\mathrm{Tr}\, B^Q$ is independent of the choice of basis!

Example:

$$\begin{aligned} \mathrm{Tr}\, \left|\psi^Q\right\rangle \left\langle\phi^Q\right| &= \sum_k \left\langle a_k^Q \middle| \psi^Q \right\rangle \left\langle \phi^Q \middle| a_k^Q \right\rangle \\ &= \sum_k \left\langle \phi^Q \middle| a_k^Q \right\rangle \left\langle a_k^Q \middle| \psi^Q \right\rangle \\ &= \left\langle \phi^Q \middle| \psi^Q \right\rangle \end{aligned}$$

since $\sum_k \left|a_k^Q\right\rangle\left\langle a_k^Q\right| = 1^Q$.

Density operators

For a pure state, $\rho^Q = \left|\psi^Q\right\rangle\left\langle\psi^Q\right|$.

- ${\rho^Q}^\dagger = \rho^Q$.
- ρ^Q is a positive operator:

  $$\left\langle\phi^Q\right| \rho^Q \left|\phi^Q\right\rangle \geq 0$$

  for all $\left|\phi^Q\right\rangle$. Thus ρ^Q has no negative eigenvalues.
- $\mathrm{Tr}\,\rho^Q = 1$.
- Given an observable A (operator A^Q),

  $$\langle A\rangle = \mathrm{Tr}\,(\rho^Q A^Q).$$

  For a pure state, this is $\left\langle\psi^Q\right| A^Q \left|\psi^Q\right\rangle$.

Ensembles

Situation: Unknown pure state

$$\left.\begin{array}{rl} \text{state} & |\phi_k\rangle \\ \text{probability} & p_k \end{array}\right\} \begin{array}{l} \text{define an} \\ \textbf{ensemble}, \\ \text{denoted by } \mathcal{M} \end{array}$$

"Mixed state" density operator for $\mathcal{M}$

$$\rho^Q = \sum_k p_k \, |\phi_k\rangle\langle\phi_k|$$

- Ensemble averages: $\langle A \rangle = \mathrm{Tr}\,(\rho^Q A^Q)$
- $\mathcal{M} \mapsto \rho^Q$ map is **many-to-one**
- "Eigen-ensemble" $\mathcal{M}_{\mathrm{eig}}$

$$\left.\begin{array}{rl} \text{eigenstates} & |\lambda_k\rangle \\ \text{eigenvalues} & \lambda_k \end{array}\right\} \quad \rho^Q = \sum_k \lambda_k \, |\lambda_k\rangle\langle\lambda_k|$$

Partial inner product

- Fix $\left|\phi^R\right\rangle$ in $\mathcal{H}^R$ and $\left|\Psi^{RQ}\right\rangle$ in $\mathcal{H}^{RQ}$. Define the partial inner product

$$\left|\chi^Q\right\rangle = \left\langle\phi^R\middle|\Psi^{RQ}\right\rangle$$

as the vector $\left|\chi^Q\right\rangle$ in $\mathcal{H}^Q$ such that

$$\left\langle\xi^Q\middle|\chi^Q\right\rangle = \left\langle\phi^R\xi^Q\middle|\Psi^{RQ}\right\rangle$$

for all $\left|\xi^Q\right\rangle$ in $\mathcal{H}^Q$.

- Let

$$\left|\phi^R\right\rangle = \sum_k d_k \left|a_k^R\right\rangle$$

$$\left|\Psi^{RQ}\right\rangle = \sum_{km} c_{km} \left|a_k^R b_m^Q\right\rangle$$

Then $\left\langle\phi^R\middle|\Psi^{RQ}\right\rangle = \sum_{km} d_k^* c_{km} \left|b_m^Q\right\rangle$

- Given an operator A^{RQ} on $\mathcal{H}^{RQ}$. Then

$$B^Q = \left\langle\phi^R\right| A^{RQ} \left|\psi^R\right\rangle$$

is an operator on $\mathcal{H}^Q$

Partial trace

Choose some orthonormal basis $\left|a_k^R\right\rangle$.

The **partial trace** of an operator A^{RQ} is

$$\mathrm{Tr}_R A^{RQ} = \sum_k \left\langle a_k^R\right| A^{RQ} \left|a_k^R\right\rangle$$

which is an operator on $\mathcal{H}^Q$.

The partial trace is basis-independent.

Example

$$\begin{aligned}
&\mathrm{Tr}_R \left|\chi^R\psi^Q\right\rangle \left\langle \xi^R\phi^Q\right| \\
&\quad = \sum_k \left\langle a_k^R \middle| \chi^R\psi^Q \right\rangle \left\langle \xi^R\phi^Q \middle| a_k^R \right\rangle \\
&\quad = \sum_k \left\langle \xi^R \middle| a_k^R \right\rangle \left\langle a_k^R \middle| \chi^R \right\rangle \left|\psi^Q\right\rangle \left\langle \phi^Q\right| \\
&\quad = \left\langle \xi^R \middle| \chi^R \right\rangle \left|\psi^Q\right\rangle \left\langle \phi^Q\right|
\end{aligned}$$

Entanglement

Situation: Pure entangled state $\left|\Psi^{RQ}\right\rangle$ of a composite system RQ

Subsystem density operator

$$\rho^Q = \mathrm{Tr}_R \underbrace{\left|\Psi^{RQ}\right\rangle\left\langle\Psi^{RQ}\right|}_{\rho^{RQ}}$$

- Q-observable averages: $\langle A\rangle = \mathrm{Tr}\,(\rho^Q A^Q)$
- $\rho^{RQ} \mapsto \rho^Q$ map is **many-to-one**
- If ρ^{RQ} is pure, then ρ^R and ρ^Q have the same non-zero eigenvalues λ_k.
- Schmidt decomposition:

$$\left|\Psi^{RQ}\right\rangle = \sum_k \sqrt{\lambda_k}\left|\lambda_k^R \lambda_k^Q\right\rangle$$

Relative states

Pure entangled state $\left|\Psi^{RQ}\right\rangle$

Fix a basis $\left|a_k^R\right\rangle$ and write

$$\begin{aligned}\left|\Psi^{RQ}\right\rangle &= \sum_{km} c_{km}\left|a_k^R b_m^Q\right\rangle \\ &= \sum_k \left|a_k^R\right\rangle \otimes \left|\tilde{\phi}_k^Q\right\rangle \\ &= \sum_k \sqrt{q_k}\left|a_k^R\right\rangle \otimes \left|\phi_k^Q\right\rangle\end{aligned}$$

$\left|\phi_k^Q\right\rangle$ are **relative states** to the $\left|a_k^R\right\rangle$

$$\begin{aligned}\rho^Q &= \mathrm{Tr}_R\left|\Psi^{RQ}\right\rangle\left\langle\Psi^{RQ}\right| \\ &= \sum_k q_k\left|\phi_k^Q\right\rangle\left\langle\phi_k^Q\right|\end{aligned}$$

$$
\begin{aligned}
\left|\Psi^{RQ}\right\rangle &= \sum_k \sqrt{q_k}\,\left|a_k^R\right\rangle \otimes \left|\phi_k^Q\right\rangle \\
\rho^Q &= \sum_k q_k \left|\phi_k^Q\right\rangle\left\langle\phi_k^Q\right|
\end{aligned}
$$

- If we measure an observable on R with eigenstates $\left|a_k^R\right\rangle$, then

probability of a_k	q_k
resulting Q-state	$\left|\phi_k^Q\right\rangle$

  "Ensemble of relative states"

- Useful fact: For a given $\left|\Psi^{RQ}\right\rangle$ we can realize **any** ρ^Q-ensemble $\mathcal{M}$ as an ensemble of relative states.

$$
\left(\begin{array}{c}\text{Different}\\ \text{ensembles}\\ \text{for } \rho^Q\end{array}\right) \Longleftrightarrow \left(\begin{array}{c}\text{Different}\\ \text{bases for } R\end{array}\right)
$$

Partial trace

- Begin with $\left|\Psi^{RQ}\right\rangle$
- Find $\rho^Q = \mathrm{Tr}_R \left|\Psi^{RQ}\right\rangle\left\langle\Psi^{RQ}\right|$
- A possible physical process!
- ρ^Q is unique

"Purification"

- Begin with ρ^Q
- Imagine $\left|\Psi^{RQ}\right\rangle$ so that $\rho^Q = \mathrm{Tr}_R \left|\Psi^{RQ}\right\rangle\left\langle\Psi^{RQ}\right|$
- A wholly imaginary process!
- $\left|\Psi^{RQ}\right\rangle$ is not unique:

$$\left|\Psi_1^{RQ}\right\rangle = \left(U_{12}^R \otimes 1^Q\right) \left|\Psi_2^{RQ}\right\rangle$$

Generalized observables

Measurement outcome k associated with "outcome operator" Π_k^Q so that

$$p_k = \mathrm{Tr}\,\rho^Q \Pi_k^Q$$

Properties of Π_k^Q

$p_k \geq 0$ for any ρ^Q	$\Longrightarrow$	Π_k^Q is a **positive** operator
$\sum_k p_k = 1$ for any ρ^Q	$\Longrightarrow$	$\sum_k \Pi_k^Q = 1^Q$

Fact: Any generalized "positive operator" observable can be modeled as a "projection operator" observable on a larger system.

Generalized time evolution

$$\rho^Q \quad \boxed{Q} \xrightarrow{\quad\mathcal{E}^Q\quad} \boxed{Q'} \quad \rho^{Q'}$$

General map $\mathcal{E}^Q : \rho^Q \mapsto \rho^{Q'}$, also known as a "superoperator".

What properties must $\mathcal{E}^Q$ have?

1. $\mathcal{E}^Q$ must be **linear**. That is,

$$\underbrace{\mathcal{E}^Q(\overbrace{p_1\rho_1^Q + p_2\rho_2^Q}^{\rho^Q})}_{\rho^{Q'}} = \underbrace{p_1\mathcal{E}^Q(\rho_1^Q) + p_2\mathcal{E}^Q(\rho_2^Q)}_{p_1\rho_1^{Q'} + p_2\rho_2^{Q'}}$$

2. $\mathcal{E}^Q$ must be **trace-preserving**, since $\operatorname{Tr}\rho^Q = \operatorname{Tr}\rho^{Q'} = 1$.

3. $\mathcal{E}^Q$ must be **positive**—that is, it must map positive operators to positive operators (since $\rho^{Q'}$ must always be positive).
4. $\mathcal{E}^Q$ must be **completely positive**.

$$\rho^{RQ} \quad \begin{array}{ccc} \boxed{R} & \xrightarrow{\quad \mathbf{I}^R \quad} & \boxed{R'} \\ & & \\ \boxed{Q} & \xrightarrow{\quad \mathcal{E}^Q \quad} & \boxed{Q'} \end{array} \quad \rho^{RQ'}$$

Suppose we add an independent system R with trivial dynamics. Then the extended superoperator $\mathbf{I}^R \otimes \mathcal{E}^Q$ must also be a positive map.

> Any reasonable quantum time evolution must be described by a trace-preserving, completely positive linear map.

Are all positive maps completely positive?

No!

Counterexample: Fix a basis $\left|a_m^Q\right\rangle$. Write

$$\rho^Q = \sum_{mn} \rho_{mn} \left|a_m^Q\right\rangle \left\langle a_n^Q\right|$$

Define the **transpose** map $\mathcal{T}^Q$ by

$$\mathcal{T}^Q(\rho^Q) = \sum_{mn} \rho_{nm} \left|a_m^Q\right\rangle \left\langle a_n^Q\right|$$

$\mathcal{T}^Q$ is a trace-preserving, positive linear map. But it is *not* completely positive.

> To prove this, append system R and apply the "partial transpose" map $\mathbf{I}^R \otimes \mathcal{T}^Q$ to an entangled pure state of RQ.

Operator-sum model

Fix a set of operators A_μ^Q and let

$$\boxed{\mathcal{E}^Q(\rho^Q) = \sum_\mu A_\mu^Q \rho^Q A_\mu^{Q\dagger}}$$

$\mathcal{E}^Q$ is a completely positive linear map.

$\mathcal{E}^Q$ will be trace-preserving if and only if the A_μ^Q operators satisfy

$$\boxed{\sum_\mu A_\mu^{Q\dagger} A_\mu^Q = 1^Q}$$

Unitary dynamics is a special case in which there is only one operator in the sum:

$$\begin{aligned} \mathcal{E}^Q(\rho^Q) &= U^Q \rho^Q U^{Q\dagger} \\ U^{Q\dagger} U^Q &= 1^Q \end{aligned}$$

Unitary model

Consider an "environment" system E initially in a pure state $|0^E\rangle$. Let

$$\boxed{\mathcal{E}^Q(\rho^Q) = \mathrm{Tr}_E U^{QE}(\rho^Q \otimes |0^E\rangle\langle 0^E|) U^{QE\dagger}}$$

where U^{QE} is unitary. Then $\mathcal{E}^Q$ is a trace-preserving, completely positive linear map.

In other words, the composite system QE undergoes unitary evolution, but E is discarded at the end.

Unitary dynamics is a special case in which there is no interaction between Q and E—that is,

$$U^{QE} = U^Q \otimes V^E$$

Connecting the pictures

Fix a unitary model of Q's evolution and specify a basis $\left|e_\mu^E\right\rangle$ for E

Define operator A_μ^Q via

$$A_\mu^Q \left|\psi^Q\right\rangle = \left\langle e_\mu^E\right| U^{QE} \left|\psi^Q 0^E\right\rangle$$

Do partial trace using $\left|e_\mu^E\right\rangle$ basis

$$\begin{aligned}
&\mathcal{E}^Q(\left|\psi^Q\right\rangle\left\langle\psi^Q\right|) \\
&= \sum_\mu \left\langle e_\mu^E\right| U^{QE} \left|\psi^Q 0^E\right\rangle\left\langle\psi^Q 0^E\right| U^{QE\dagger} \left|e_\mu^E\right\rangle \\
&= \sum_\mu A_\mu^Q \left|\psi^Q\right\rangle\left\langle\psi^Q\right| A_\mu^{Q\dagger}
\end{aligned}$$

Choice of $\left|e_\mu^E\right\rangle$ basis $\Longrightarrow$ Choice of operator-sum model

Representation theorem

- Every trace-preserving completely positive linear map $\mathcal{E}^Q$ has an operator-sum representation

$$\mathcal{E}^Q(\rho^Q) = \sum_\mu A^Q_\mu \rho^Q A^{Q\,\dagger}_\mu$$

where $\sum_\mu A^{Q\,\dagger}_\mu A^Q_\mu = 1^Q$ (normalization) .

- Every trace-preserving completely positive linear map $\mathcal{E}^Q$ has a unitary representation

$$\mathcal{E}^Q(\rho^Q) = \mathrm{Tr}_E U^{QE}(\rho^Q \otimes |0^E\rangle\langle 0^E|) U^{QE\,\dagger}.$$

(In neither case is the representation unique.)

Some remarks:

- Unitary representations introduce an auxilliary system E. Operator-sum representations are "intrinsic" to Q.
- The "environment" system E need not be the actual environment.
- Physically interesting quantities should not depend on what representation is chosen for $\mathcal{E}^Q$.

A sketch of the proof

1. Begin with a completely positive, trace-preserving linear map $\mathcal{E}^Q$.

2. It suffices to consider pure state inputs $\rho^Q = \left|\psi^Q\right\rangle\left\langle\psi^Q\right|$ (linearity).

3. Add auxilliary system R with trivial dynamics. $\mathbf{I}^R \otimes \mathcal{E}^Q$ is a positive map.

4. Consider an entangled input state

$$\left|\tilde{\Psi}^{RQ}\right\rangle = \sum_k \left|a_k^R b_k^Q\right\rangle$$

5. Apply $\mathcal{E}^Q$

$$\tilde{D}^{RQ'} = \mathcal{E}^Q\left(\left|\tilde{\Psi}^{RQ})\right\rangle\left\langle\tilde{\Psi}^{RQ})\right|\right)$$

$\tilde{D}^{RQ'}$ is a positive operator.

6. Index states

   For every $\left|\phi^Q\right\rangle$ we can find a state $\left|\phi^{*R}\right\rangle$ so that

$$\left|\phi^Q\right\rangle = \left\langle \phi^{*R} \middle| \tilde{\Psi}^{RQ} \right\rangle$$

   The relation of $\left|\phi^Q\right\rangle$ to $\left|\phi^{*R}\right\rangle$ is conjugate-linear. That is,

$$\begin{aligned} \text{if } \left|\phi^Q\right\rangle &= c_1 \left|\phi_1^Q\right\rangle + c_2 \left|\phi_2^Q\right\rangle \\ \text{then } \left|\phi^{*R}\right\rangle &= c_1^* \left|\phi_1^{*R}\right\rangle + c_2^* \left|\phi_2^{*R}\right\rangle \end{aligned}$$

   Clearly $\rho^Q = \left\langle \phi^{*R} \middle| \tilde{\Psi}^{RQ} \right\rangle \left\langle \tilde{\Psi}^{RQ} \middle| \phi^{*R} \right\rangle$.

7. Applying the index state commutes with $\mathcal{E}^Q$, since they are operations on different systems. Thus

$$\mathcal{E}^Q(\rho^Q) = \left\langle \phi^{*R} \right| \tilde{D}^{RQ'} \left| \phi^{*R} \right\rangle$$

8. Write $\tilde{D}^{RQ'} = \sum_\mu \left|\tilde{\mu}^{RQ'}\right\rangle\left\langle\tilde{\mu}^{RQ'}\right|$.

9. Define A_μ^Q by

$$A_\mu^Q \left|\phi^Q\right\rangle = \left\langle \phi^{*R} \middle| \tilde{\mu}^{RQ'} \right\rangle$$

and show

$$\mathcal{E}^Q(\rho^Q) = \sum_\mu A_\mu^Q \rho^Q A_\mu^{Q\dagger}$$

10. Verify that $\sum_\mu A_\mu^{Q\dagger} A_\mu^Q = 1^Q$. This relies on the fact that $\mathcal{E}^Q$ is trace-preserving.

11. Choose a basis $\left|e_\mu^E\right\rangle$ for E and let

$$\left|\tilde{\Upsilon}^{RQE'}\right\rangle = \sum_\mu \left|\tilde{\mu}^{RQ'}\right\rangle \otimes \left|e_\mu^E\right\rangle$$

12. For a given $\left|\phi^Q\right\rangle$, define

$$\left|\phi^{QE'}\right\rangle = \left\langle\phi^{*R}\middle|\tilde{\Upsilon}^{RQE'}\right\rangle$$

13. Show that

$$\mathcal{E}^Q(\rho^Q) = \mathrm{Tr}_E \left|\phi^{QE'}\right\rangle\left\langle\phi^{QE'}\right|$$

14. Fix a state $\left|0^E\right\rangle$ of E. Show that the (linear) map

$$\left|\phi^Q 0^E\right\rangle \longrightarrow \left|\phi^{QE'}\right\rangle$$

preserves inner products. Extend it to a unitary operator U^{RQ} so that

$$\mathcal{E}^Q(\rho^Q) = \mathrm{Tr}_E U^{QE}(\rho^Q \otimes \left|0^E\right\rangle\left\langle 0^E\right|) U^{QE\dagger}.$$

“No-cloning” theorem

Can we make a perfect quantum Xerox machine?

$$\begin{array}{ccc} \left|\psi^{Q_1}\right\rangle \otimes \left|0^{Q_2}\right\rangle & \longrightarrow & \left|\psi^{Q_1}\right\rangle \otimes \left|\psi^{Q_2}\right\rangle \\ \text{original} + \text{blank} & \longrightarrow & \text{two copies} \end{array}$$

The representation theorem allows us to suppose that this process is unitary on a larger system:

$$\left|\psi^{Q_1}\psi^{Q_2}\Phi^{E'}\right\rangle = U^{Q_1 Q_2 E}\left|\psi^{Q_1} 0^{Q_2} \Phi_0^{E}\right\rangle$$

where $\left|\Phi^{E'}\right\rangle$ may depend on $\left|\psi^{Q_1}\right\rangle$.

Consider two possible inputs $\left|\psi_1^{Q_1}\right\rangle$ and $\left|\psi_2^{Q_1}\right\rangle$. Since the overall process is unitary,

$$
\begin{aligned}
\left\langle\psi_1^{Q_1}\middle|\psi_2^{Q_1}\right\rangle &= \left\langle\psi_1^{Q_1}0^{Q_2}\Phi_0^{E}\middle|\psi_2^{Q_1}0^{Q_2}\Phi_0^{E}\right\rangle \\
&= \left\langle\psi_1^{Q_1}\psi_1^{Q_2}\Phi_1^{E'}\middle|\psi_2^{Q_1}\psi_2^{Q_2}\Phi_2^{E'}\right\rangle \\
&= \left(\left\langle\psi_1^{Q_1}\middle|\psi_2^{Q_1}\right\rangle\right)^2\left\langle\Phi_1^{E'}\middle|\Phi_2^{E'}\right\rangle
\end{aligned}
$$

This could only be true in two cases:

1. $\left|\psi_1^{Q_1}\right\rangle = e^{i\varphi}\left|\psi_2^{Q_1}\right\rangle$, so that $\left|\left\langle\psi_1^{Q_1}\middle|\psi_2^{Q_1}\right\rangle\right| = 1$
2. $\left\langle\psi_1^{Q_1}\middle|\psi_2^{Q_1}\right\rangle = 0$, so that $\left|\psi_1^{Q_1}\right\rangle$ and $\left|\psi_2^{Q_1}\right\rangle$ can be perfectly distinguished by a measurement.

No reasonable process can clone an arbitrary quantum state.

The Church
of the Larger Hilbert Space

Creed

- Every state can be made pure
- Every measurement can be made ideal
- Every evolution can be made unitary

Sacramental rite

Purification

Principle of Harmony

Different people may purify in different ways, but they nevertheless arrive at the same results.